Math 4FM3, Tutorial 1.

1. Binomial Option Pricing Models.

σ = Annual volatility = Annualized standard deviation of the continuously compounded stock return.

① The Standard Binomial Model

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}, \quad d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

h is time increment, r is risk-free rate. δ is dividend.

Ex 3-month 3-step binomial tree

The forward price $F_{t,t+h} = S_t e^{(r-\delta)h}$, then

$$uS_t = S_t e^{(r-\delta)h+\sigma\sqrt{h}} = F_{t,t+h} e^{\sigma\sqrt{h}}, \quad dS_t = S_t e^{(r-\delta)h-\sigma\sqrt{h}} = F_{t,t+h} e^{-\sigma\sqrt{h}}.$$

~~Under this model~~

② The Cox-Ross-Rubinstein Binomial Tree.

$u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$, (matching volatility with u and d)

B.7

We need to select parameters such that $e^{(r-\delta)h} < e^{\sigma\sqrt{h}}$

In practice, h is quite small, then $u_{CRR} \approx u_{standard}$.

2. Current price is 60, not dividend. The volatility $\sigma = 0.3$. $r = 0.08$, strike price $K = 58$, Find 1-year European call option.

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.08+0.3} = 1.4623, \Rightarrow S_u = uS = 87.737$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.08-0.3} = 0.8025 \Rightarrow S_d = dS = 48.151.$$

Stock $\quad\quad\quad\quad\quad\quad\quad$ Option

$$60 \begin{cases} 87.737 \\ 48.151 \end{cases} \qquad V \begin{cases} 29.737 \\ 0 \end{cases}$$

The option can be replicated by buying Δ shares of stock and lending B dollars at risk-free rate. (bonds)

$$\Delta = e^{-\delta h} \cdot \frac{V_u - V_d}{S(u-d)} = 0.7512.$$

$$B = e^{-rh} \cdot \frac{uV_d - dV_u}{u-d} = -33.390.$$

The cost of option is equal to buying 0.7512 shares and borrowing 33.390.

$$V = \Delta S + B = 11.682.$$

If the probability of up state is $p = 54.46\%$, then the expected return is

$$e^{\alpha} = \frac{pS_u - qS_d}{S} \Rightarrow \alpha = 0.15, \text{ for stock}$$

$$e^{\gamma} = \frac{pV_u - qV_d}{V} \Rightarrow \gamma = 0.3267 \text{ for option.}$$

Note: α = expected return $\neq$ expected value of rate of return.

Also, suppose $p = 72.98\%$, then $\alpha = 0.25$, $\gamma = 0.6194$.

suppose $p = p^*$ (risk-neutral) $= \dfrac{e^{(r-\delta)h} - d}{u-d} = 0.4256.$

$\Rightarrow \alpha = \gamma = 0.08 = r.$

If we are given p, α, and γ, then S and V can be also be calculated. $S = 60$, $V = 11.687$.

Replication can be used to find the price of an option but this price does not depend on the p assigned by market participants.

3. Options on Currencies.

A foreign currency can be regarded as an asset providing a yield at the foreign risk-free rate of interest r_f.

Ex 13.2.

1 AUD worth 0.61 USD. Volatility of exchange rate $\sigma = 0.12$

Australian risk-free rate $r_f = 0.07$,

US risk-free rate $r = 0.05$.

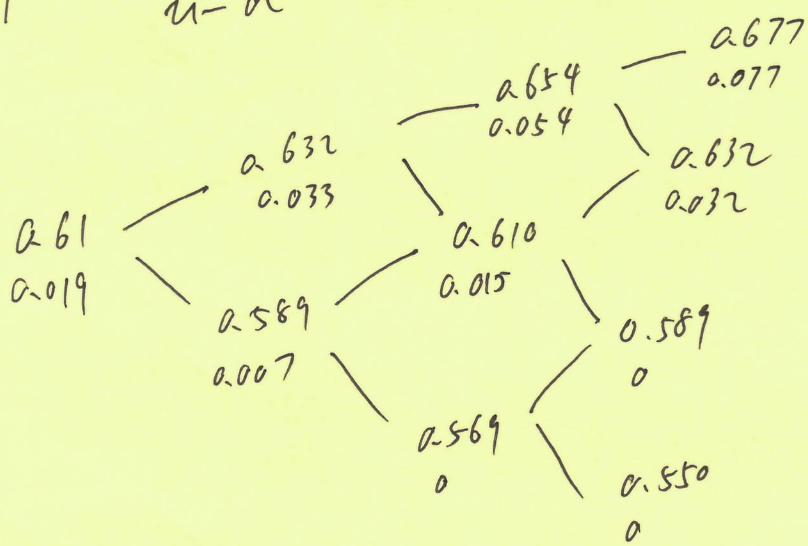
For a 3-month American call option with $K = 0.6$ in 3-step binomial tree

$S_0 = 0.61$

$\Delta t = \cancel{0.08333}\ \frac{1}{12} = 0.08333$

$u = e^{\sigma\sqrt{\Delta t}} = 1.0352$, $\quad d = \frac{1}{u} = 0.9660$.

$p = \dfrac{e^{(r-r_f)\Delta t} - d}{u-d} = 0.4673$.



4. Option on Futures.

It costs nothing to take a long or a short position in a future contract.

It follows that in a risk-neutral world a future price should have an expected growth rate of zero.

$p F_0 u + (1-p) F_0 d = F_0 \Rightarrow p = \dfrac{1-d}{u-d}$.

Ex 13.3.

A future price is currently 31 with volatility 30%, $r = 5\%$.

For a 9-month American put option with $K = 30$. in 3-step binomial tree

$\Delta t = \frac{9/3}{12} = 0.25$, $u = e^{0.3\sqrt{0.25}} = 1.1618$, $d = 1/u = 0.8607$,

$p = \frac{1-d}{u-d} = 0.4626$.